DIFFERENTIAL GEOMETRY 100 POINTS

Notes.
(a) You may freely use any result proved in class. All other steps must be justified.
(b) $\mathbb{R}$ = real numbers.

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1. [16 points] Let p be a point on a regular curve $\gamma(t)$ in $\mathbb{R}^3$ with $\gamma(0) = p$. Let $\alpha(t)$ be the curve resulting from the orthogonal projection of $\gamma(t)$ to the osculating plane to $\gamma(t)$ at p. Verify that $\alpha(t)$ is also regular at $t = 0$ and prove that the curvatures of the two curves are equal at p.
(*Hint.* Use translation and rotation to simplify.)

2. [16 points] Suppose $\alpha(s)$ and $\beta(s)$ are two unit-speed curves in $\mathbb{R}^3$ through a point p with $\alpha(0) = p = \beta(0)$ and having positive torsion everywhere. If the associated binormal vectors $B_\alpha(s)$ and $B_\beta(s)$ are equal for all s, then prove that $\alpha(s) = \beta(s)$.

3. [20 points] Let $F(x, y, z) = (z + e^{y+\cos x},\, y + \sin x,\, x^5 + x)$. Prove that $F\colon R^3 \to R^3$ is a diffeomorphism.

4. [24 points] Let S_1 denote the finite cylinder $x^2 + y^2 = 1, 1 < z < 2$ in $\mathbb{R}^3$, S_2 the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ and S_3 the Mobius strip in $\mathbb{R}^3$ covered by suitable patches parametrized by

$$\sigma(t, \theta) = \left(\left(1 - t\sin\frac{\theta}{2}\right)\cos\theta,\ (1 - t\sin\frac{\theta}{2})\sin\theta,\ t\cos\frac{\theta}{2}\right),$$

using $|t| < 1/2, |\theta| < \pi$ for one patch and $|t| < 1/2, 0 < \theta < 2\pi$ for the other. Prove or disprove the following:

(i) There is a diffeomorphism from S_1 to an open subset of S_2.
(ii) There is a diffeomorphism from S_3 to an open subset of S_2.
(iii) For any $p \in S_1$ and any $q \in S_2$, there is a conformal diffeomorphism from an open neighbourhood of p in S_1 to an open neighbourhood of q in S_2.

5. [24 points] Let S be the surface of revolution obtained by rotating about the z-axis, the profile curve in the x-z plane given by $(x - 2)^2 + (z/2)^2 = 1$.

(i) Calculate the first and second fundamental forms, the Weingarten matrix, the principal normal curvatures and the Gaussian curvature at all the points of S.
(ii) At $p = (2, 0, 2) \in S$, find the normal curvatures of S along the unit tangent vector $w = (\cos\theta, \sin\theta, 0)$.